

A refinement of Ramanujan's factorial approximation

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1 Introduction

All we have of Ramanujan's work in the last year of his life is about 100 pages (probably a small fraction of his final year's output), held by Trinity College, Cambridge, and named by George E. Andrews "Ramanujan's Lost Notebook". It was published in photocopied form [3]. In it, Ramanujan [3, p. 339] makes the claim that

$$\Gamma(x+1) = \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right)^{\frac{1}{6}}$$

where $\theta_x \rightarrow 1$ as $x \rightarrow \infty$ and $\frac{3}{10} < \theta_x < 1$, and gives some numerical evidence for this last statement.

Inspired by this, we confine ourselves to the positive integers, and prove the following stronger result.

Theorem 1. *Let the function, $\theta(n) \equiv \theta_n$, be defined for $n = 1, 2, \dots$ by the equation:*

$$n! := \sqrt{\pi} \left(\frac{n}{e}\right)^n \left(8n^3 + 4n^2 + n + \frac{\theta(n)}{30}\right)^{\frac{1}{6}}.$$

Then, the correction term $\theta(n)$

(a) satisfies the inequalities:

$$1 - \frac{11}{8n} + \frac{79}{112n^2} < \theta(n) < 1 - \frac{11}{8n} + \frac{79}{112n^2} + \frac{20}{33n^3}; \quad (1)$$

*(b) is an **increasing** function of n ; and*

*(c) is **concave**, that is,*

$$\theta_{n+1} - \theta_n < \theta_n - \theta_{n-1}.$$

The inequalities (1) are new. In 2006, Hirschhorn [1] proved a less exact version of the inequalities (1). In 2001, Karatsuba [2] proved Ramanujan's approximation and gave a proof, quite different from ours, of the monotonicity of the correction

term θ_x , for all real $x \geq 1$, a result which is stronger than ours. Moreover, although Karatsuba derived an asymptotic expansion for θ_x , including a uniform error term, she did not derive any explicit numerical inequalities, as we do. The monotonicity of $\theta(n)$ was proved by Villarino, Campos-Salas, and Carvajal-Rojas in [4] as a simple consequence of the inequality in [1]; in that paper, the concavity of $\theta(n)$ was also noted, without proof.

Our proofs use nothing more than the series for $\log(1+x)$ and $\exp\{x\}$.

We will find it convenient to use the following notation.

Definition 2. *The notation*

$$P_k(n)$$

*means a polynomial of degree k in n with all of its non-zero coefficients **positive**.*

2 The proofs

Proposition 3. *The following inequality is valid for $n = 1, 2, \dots$:*

$$0 < \left(n + \frac{1}{2}\right) \log\left(1 + \frac{1}{n}\right) - 1 < \frac{1}{12n} - \frac{1}{12(n+1)}. \quad (2)$$

Proof. We have, for $|x| < 1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots.$$

It follows that for $|x| < 1$,

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right).$$

If we set $x = \frac{1}{2n+1}$ where $n \in \mathbb{Z}^+$, we obtain

$$\log\left(1 + \frac{1}{n}\right) = 2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\right). \quad (3)$$

It follows that

$$\left(n + \frac{1}{2}\right) \log\left(1 + \frac{1}{n}\right) = 1 + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \dots$$

Therefore

$$0 < \left(n + \frac{1}{2}\right) \log\left(1 + \frac{1}{n}\right) - 1 < \frac{1}{3(2n+1)^2} \cdot \frac{1}{1 - \frac{1}{(2n+1)^2}} = \frac{1}{12n} - \frac{1}{12(n+1)}. \quad \square$$

The inequality (2) leads to the following well-known version of Stirling's formula.

Proposition 4. *The following inequality is valid for $n = 1, 2, \dots$:*

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n < n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left\{\frac{1}{12n}\right\}. \quad (4)$$

Proof. Let

$$a_n = n! / \sqrt{n} \left(\frac{n}{e} \right)^n.$$

Then

$$\frac{a_n}{a_{n+1}} = \left(1 + \frac{1}{n} \right)^{n+\frac{1}{2}} / e = \exp \left\{ \left(n + \frac{1}{2} \right) \log \left(1 + \frac{1}{n} \right) - 1 \right\}.$$

From (2) we have

$$1 < \frac{a_n}{a_{n+1}} < \exp \left\{ \frac{1}{12n} - \frac{1}{12(n+1)} \right\}. \quad (5)$$

So a_n is decreasing and, if we write $1, 2, \dots, n-1$ for n and multiply the results, we find

$$\frac{a_1}{a_n} < \exp \left\{ \frac{1}{12} - \frac{1}{12n} \right\} < \exp \left\{ \frac{1}{12} \right\},$$

or,

$$a_n > a_1 \exp \left\{ -\frac{1}{12} \right\} = \exp \left\{ \frac{11}{12} \right\}.$$

It follows that $a_\infty = \lim_{n \rightarrow \infty} a_n$ exists, and

$$a_\infty \geq \exp \left\{ \frac{11}{12} \right\}.$$

In fact, from Wallis's product, $a_\infty = \sqrt{2\pi}$.

If in (5) we write $n, n+1, \dots, N-1$ for n , multiply the results, let $N \rightarrow \infty$ and use Wallis's product, we obtain, successively,

$$\begin{aligned} 1 &< \frac{a_n}{a_N} < \exp \left\{ \frac{1}{12n} - \frac{1}{12N} \right\}, \\ a_N &< a_n < a_N \exp \left\{ \frac{1}{12n} - \frac{1}{12N} \right\}, \\ \sqrt{2\pi} &< a_n \leq \sqrt{2\pi} \exp \left\{ \frac{1}{12n} \right\} \end{aligned}$$

and

$$\sqrt{2\pi n} \left(\frac{n}{e} \right)^n < n! \leq \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \exp \left\{ \frac{1}{12n} \right\}. \quad \square$$

We shall improve on (2), and so also on (4), by proving the next inequality.

Proposition 5. *The following inequality is valid for $n = 1, 2, \dots$:*

$$\begin{aligned} &\frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1} \right) - \frac{1}{360} \left(\frac{1}{n^3} - \frac{1}{(n+1)^3} \right) + \frac{1}{1260} \left(\frac{1}{n^5} - \frac{1}{(n+1)^5} \right) \\ &\quad - \frac{1}{1680} \left(\frac{1}{n^7} - \frac{1}{(n+1)^7} \right) \\ &< \left(n + \frac{1}{2} \right) \log \left(1 + \frac{1}{n} \right) - 1 \\ &< \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1} \right) - \frac{1}{360} \left(\frac{1}{n^3} - \frac{1}{(n+1)^3} \right) + \frac{1}{1260} \left(\frac{1}{n^5} - \frac{1}{(n+1)^5} \right). \quad (6) \end{aligned}$$

Proof. We have, from (3),

$$\begin{aligned}
\left(n + \frac{1}{2}\right) \log\left(1 + \frac{1}{n}\right) - 1 &< \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \frac{1}{7(2n+1)^6} \cdot \frac{1}{1 - \frac{1}{(2n+1)^2}} \\
&= \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \frac{1}{28n(n+1)(2n+1)^4} \\
&= \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{360} \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \frac{1}{1260} \left(\frac{1}{n^5} - \frac{1}{(n+1)^5}\right) \\
&\quad - \frac{163n^6 + 489n^5 + 604n^4 + 393n^3 + 141n^2 + 26n + 2}{2520n^5(n+1)^5(2n+1)^4} \\
&< \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{360} \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \frac{1}{1260} \left(\frac{1}{n^5} - \frac{1}{(n+1)^5}\right)
\end{aligned}$$

and

$$\begin{aligned}
\left(n + \frac{1}{2}\right) \log\left(1 + \frac{1}{n}\right) - 1 &> \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \frac{1}{7(2n+1)^6} + \frac{1}{9(2n+1)^8} \\
&= \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{360} \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \frac{1}{1260} \left(\frac{1}{n^5} - \frac{1}{(n+1)^5}\right) \\
&\quad - \frac{1}{1680} \left(\frac{1}{n^7} - \frac{1}{(n+1)^7}\right) \\
&\quad + \frac{P_{12}(n)}{5040n^7(n+1)^7(2n+1)^8} \\
&> \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{360} \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \frac{1}{1260} \left(\frac{1}{n^5} - \frac{1}{(n+1)^5}\right) \\
&\quad - \frac{1}{1680} \left(\frac{1}{n^7} - \frac{1}{(n+1)^7}\right).
\end{aligned}$$

This completes the proof. \square

We now demonstrate the greatly improved version of Stirling's formula.

Proposition 6. *For $n = 1, 2, \dots$, the following inequality is valid:*

$$\begin{aligned}
&\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left\{\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7}\right\} \\
&\leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left\{\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5}\right\}.
\end{aligned} \tag{7}$$

Proof. It follows from (6) that

$$\begin{aligned}
&\exp\left\{\frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{360} \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \frac{1}{1260} \left(\frac{1}{n^5} - \frac{1}{(n+1)^5}\right) \right. \\
&\quad \left. - \frac{1}{1680} \left(\frac{1}{n^7} - \frac{1}{(n+1)^7}\right)\right\} \\
&< \frac{a_n}{a_{n+1}} \\
&< \exp\left\{\frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{360} \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \frac{1}{1260} \left(\frac{1}{n^5} - \frac{1}{(n+1)^5}\right)\right\}.
\end{aligned}$$

Thus, for $N > n$,

$$\begin{aligned} & \exp\left\{\frac{1}{12}\left(\frac{1}{n} - \frac{1}{N}\right) - \frac{1}{360}\left(\frac{1}{n^3} - \frac{1}{N^3}\right) + \frac{1}{1260}\left(\frac{1}{n^5} - \frac{1}{N^5}\right) - \frac{1}{1680}\left(\frac{1}{n^7} - \frac{1}{N^7}\right)\right\} \\ & < \frac{a_n}{a_N} < \exp\left\{\frac{1}{12}\left(\frac{1}{n} - \frac{1}{N}\right) - \frac{1}{360}\left(\frac{1}{n^3} - \frac{1}{N^3}\right) + \frac{1}{1260}\left(\frac{1}{n^5} - \frac{1}{N^5}\right)\right\}, \end{aligned}$$

and

$$\begin{aligned} & \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left\{\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7}\right\} \\ & \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left\{\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5}\right\}. \quad \square \end{aligned}$$

Extracting the fraction $\frac{1}{6}$ from the exponents, we see that we can write this last inequality in the form

$$\begin{aligned} & \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(\exp\left\{\frac{1}{2n} - \frac{1}{60n^3} + \frac{1}{210n^5} - \frac{1}{280n^7}\right\}\right)^{\frac{1}{6}} \\ & \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(\exp\left\{\frac{1}{2n} - \frac{1}{60n^3} + \frac{1}{210n^5}\right\}\right)^{\frac{1}{6}}. \end{aligned}$$

We now obtain upper and lower bounds for these new exponents.

Proposition 7. *The following inequalities are valid for $n \geq 2$:*

$$\begin{aligned} & \exp\left\{\frac{1}{2n} - \frac{1}{60n^3} + \frac{1}{210n^5} - \frac{1}{280n^7}\right\} \\ & > 1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{240n^3} - \frac{11}{1920n^4} + \frac{79}{26880n^5} \end{aligned} \quad (8)$$

and

$$\begin{aligned} & \exp\left\{\frac{1}{2n} - \frac{1}{60n^3} + \frac{1}{210n^5}\right\} \\ & < 1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{240n^3} - \frac{11}{1920n^4} + \frac{79}{26880n^5} + \frac{1}{396n^6}. \end{aligned} \quad (9)$$

Assuming for the moment that these bounds are valid, we can now prove the main result of this paper.

Proof of Theorem 1. It follows from Proposition 7 that for $n \geq 2$,

$$\begin{aligned} & \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{240n^3} - \frac{11}{1920n^4} + \frac{79}{26880n^5}\right)^{\frac{1}{6}} \\ & < n! < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{240n^3} - \frac{11}{1920n^4} + \frac{79}{26880n^5} + \frac{1}{396n^6}\right)^{\frac{1}{6}}, \end{aligned}$$

or:

$$\begin{aligned} & \sqrt{\pi} \left(\frac{n}{e} \right)^n \left(8n^3 + 4n^2 + n + \frac{1}{30} \left(1 - \frac{11}{8n} + \frac{79}{112n^2} \right) \right)^{\frac{1}{6}} \\ & < n! < \sqrt{\pi} \left(\frac{n}{e} \right)^n \left(8n^3 + 4n^2 + n + \frac{1}{30} \left(1 - \frac{11}{8n} + \frac{79}{112n^2} + \frac{20}{33n^3} \right) \right)^{\frac{1}{6}}. \end{aligned}$$

This beautiful formula is the refined estimate (1). It is easy to check this for $n = 1$ also, so we have the desired result.

To show that $\theta(n)$ is increasing, from (1) it follows that

$$\begin{aligned} \theta_{n+1} - \theta_n &> 1 - \frac{11}{8(n+1)} + \frac{79}{112(n+1)^2} - \left(1 - \frac{11}{8n} + \frac{79}{112n^2} + \frac{20}{33n^3} \right) \\ &= \frac{(5082n^2 + 7792n + 8497)(n-2) + 14754}{3696n^3(n+1)^2} \\ &> 0 \quad \text{for } n \geq 2, \end{aligned}$$

and it is easily checked for $n = 1$ also, so θ_n is increasing.

Finally, to prove the concavity of $\theta(n)$, we note that:

$$\begin{aligned} & \theta_{n+1} - 2\theta_n + \theta_{n-1} \\ & < 1 - \frac{11}{8(n+1)} + \frac{79}{112(n+1)^2} + \frac{20}{33(n+1)^3} \\ & \quad + 1 - \frac{11}{8(n-1)} + \frac{79}{112(n-1)^2} + \frac{20}{33(n-1)^3} - 2 \left(1 - \frac{11}{8n} + \frac{79}{112n^2} \right) \\ & = - \frac{(2842n^4 + 6389n^3 + 15061n^2 + 85733n + 433747)(n-5) + 2166128}{1848n^2(n-1)^3(n+1)^3} \\ & < 0 \quad \text{for } n \geq 5, \end{aligned}$$

and is easily checked for $n = 2, 3$ and 4 also. □

We complete the proof of the exponential inequalities as follows.

Proof of Proposition 7. Let $q := \frac{1}{2n} - \frac{1}{60n^3} + \frac{1}{210n^5} - \frac{1}{280n^7}$. Then $q > 0$, and

$$\begin{aligned} \exp\{q\} &> 1 + \frac{q}{1!} + \frac{q^2}{2!} + \frac{q^3}{3!} + \frac{q^4}{4!} + \frac{q^5}{5!} \\ &= 1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{240n^3} - \frac{11}{1920n^4} + \frac{79}{26880n^5} \\ & \quad + \frac{P_{28}(n)(n-2) + 5421638789368547485949}{50185433088000000n^{35}} \\ &> 1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{240n^3} - \frac{11}{1920n^4} + \frac{79}{26880n^5} \end{aligned}$$

which proves (8) for $n \geq 2$. Now let $r := \frac{1}{2n} - \frac{1}{60n^3} + \frac{1}{210n^5}$. Then $r > 0$, and

$$\begin{aligned} \exp\{r\} &< 1 + \frac{r}{1!} + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \frac{r^5}{5!} + \frac{r^6}{6!} + \frac{r^7}{6!} + \cdots \\ &= 1 + \frac{r}{1!} + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \frac{r^5}{5!} + \frac{r^6}{6!} \Big/ (1-r) \\ &= 1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{240n^3} - \frac{11}{1920n^4} + \frac{79}{26880n^5} + \frac{1}{396n^6} \\ &\quad - \frac{P_{23}(n)(n-3) + 239259521624400145687307843}{20701491148800000n^{25}(420n^5 - 210n^4 + 7n^2 - 2)} \\ &< 1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{240n^3} - \frac{11}{1920n^4} + \frac{79}{26880n^5} + \frac{1}{396n^6} \quad \text{for } n \geq 3, \end{aligned}$$

which proves (9) for $n \geq 3$. The case $n = 2$ is easily checked. \square

3 Final Remarks

Proposition 4 and *Proposition 6* are special cases of the general expansion of $n!$, with an error term, which can be proved by using the Euler–Maclaurin sum formula. However, our proofs are much more elementary, and can be extended to any degree of accuracy desired. Still, our proofs do not supply the general formula for the coefficients in the exponential version, although perhaps they can be properly modified to do so.

Also, our technique for proving the positivity of certain large degree polynomials seems to argue for a general property of polynomials $P(x)$ with real coefficients that are positive for $x \geq a$. The property in question is that there exists a $b \geq a$ such that the quotient polynomial $Q(x)$ in the division algorithm $P(x) \equiv Q(x)(x-b) + R$ has all its coefficients positive.

Finally, the inequalities (1) can, with more work, be extended to degrees three, four, etc., where the main coefficients are given by the formula of Karatsuba [2].

We also conjecture that the correction term θ_n is *completely* monotonic.

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